Lexicographic Inference for Partially Ordered Belief Bases

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A Lexicographic Inference for Partially Preordered Belief Bases

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Abstract

Coherence-based approaches are quite popular to reason under inconsistency. Most of them are defined with respect to totally preordered belief bases such as the lexicographic inference which is known to have desirable properties from theoretical, practical and psychological points of view. However, partially preordered belief bases offer much more flexibility to represent efficiently incomplete knowledge and to avoid comparing unrelated pieces of information. In this paper, we propose a lexicographic inference for partially preordered belief bases that extends the classical one. On one hand, we define a natural inference relation which consists in applying classical lexicographic inference from all compatible totally preordered belief bases. On the other hand, we propose a novel cardinality-based preorder between consistent subbases. This cardinality-based preorder can be indifferently applied on partially or totally preordered belief bases. Then, applying classical inference on the preferred consistent subbases, according to this preorder, provides another lexicographic inference relation for partially preordered belief bases. Interestingly enough, we show that the second inference is covered by the first one. Lastly, a semantic characterization of these two definitions is provided.

Keywords partially preordered belief bases, lexicographic inference, reasoning under inconsistency.

Introduction

In Artificial Intelligence, reasoning under inconsistency represents a fundamental problem that arises in many situations such as exceptions tolerant reasoning, belief revision, integrating pieces of information coming from different possibly conflicting sources, reasoning with uncertainty or from incomplete information, etc. In this case, classical inference cannot be directly used since from an inconsistent base every formula can be inferred (ex falso quodlibet sequitur principle).

Many approaches have been proposed in order to reason under inconsistency without trivialization. While some of them consist in weakening the inference relation such as paraconsistent logics (da Costa 1974), others weaken the available beliefs like the so-called coherence-based approaches which are quite popular.

Most of the coherence-based approaches (Resher and Manor 1970) are defined with respect to totally preordered belief bases like the possibilistic inference (Dubois, Lang, and Prade 1994) or adjustment revision (Williams 1995), the linear-based revision (Nebel 1994), the inclusion-preference inference (Brewka 1989), the preference-based default reasoning (Ritterskamp and Kern-Isberner 2008) and the lexicographic inference (Benferhat et al. 1993; Lehmann 1995).

Partially preordered belief bases offer much more flexibility in order to efficiently represent incomplete knowledge and to avoid comparing unrelated pieces of information. Indeed, in many applications, the priority relation associated with available beliefs is only partially defined and forcing the user to introduce additional unwanted priorities may lead to infer undesirable conclusions.

Naturally, the flexibility and the efficiency offered by partially preordered belief bases have motivated the definition of new coherence-based approaches dedicated to partially preordered belief bases by extending the ones defined initially in the case of totally preordered belief bases. For instance, one can list the extension of the possibilistic inference proposed in (Benferhat, Lagrue, and Papini 2004) and the extensions of the inclusion-preference inference given in (Junker and Brewka 1989).

Surprising enough, no extension has been proposed for the lexicographic inference despite the fact that it is known to be satisfactory from theoretical and practical points of view. Indeed, it has been shown to be more productive than both the possibilistic and the inclusion-preference inferences. For example, it does not suffer from the drowning effect like the possibilistic inference. Moreover, according to a psychological study achieved in (Benferhat, Bonnefon, and Da Silva Neves 2004) in the context of default reasoning, it has been proved to be the most interesting among the other inference relations considered in the same study.

Our main interest in this paper is the definition of a lexicographic inference for partially preordered belief bases. We first define a natural and intuitive inference relation which consists in applying classical lexicographic inference from all the totally preordered belief bases which are compatible with the partially belief base at hand. Then, we propose a novel cardinality-based or lexicographic preorder over consistent subbases derived from the partially preordered belief base in question. Applying classical inference only on the preferred consistent subbases according to this preorder provides another lexicographic inference relation for partially preordred belief bases. The good news is that the conclusions yielded by the second inference relation are also conclusions of the first natural inference relation and both extend the classical lexicographic inference from totally preordered belief bases. Lastly, a semantic characterization of these two t definitions is provided. All proofs are provided in the Appendix.

Background

Notations

We consider a finite set of propositional variables V where its elements are denoted by lower case Roman letters $a, b, \ldots$ The symbols $\top$ and $\bot$ denote tautology and contradiction respectively. Let PL_V be the propositional language built from V, $\{\top, \bot\}$ and the connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ in the usual way. Formulae, i.e., elements of PL_V are denoted by Greek letters $\phi, \varphi, \psi, \ldots$ The set of formulae are denoted by the upper case Roman letters $A, B, \ldots$ The symbol $\vdash$ denotes classical inference relation. Let Σ be a finite set of formulae, $CONS(\Sigma)$ denotes the set of all consistent subbases of Σ while $MCONS(\Sigma)$ denotes the set of all maximally consistent subbases of Σ. Namely, $A \in MCONS(\Sigma)$ if and only if $A \subseteq \Sigma$, A is consistent and there is no $B \supsetneq A$ such that B is consistent. If Σ is consistent, then $MCONS(\Sigma)$ contains one element which is Σ. The set of interpretations is denoted by $\mathcal{W}$ and ω denotes an element of $\mathcal{W}$. Let ϕ be a propositional formula, $Mod(\phi)$ denotes the set of models of ϕ, namely $Mod(\phi) = \{\omega \in \mathcal{W} : \omega \models \phi\}$.

A partial preorder $\preceq$ on a finite set A is a reflexive and transitive binary relation. In this paper, $a \preceq b$ expresses that a is at least as preferred as b. A strict order $\prec$ on A is an irreflexive and transitive binary relation. $a \prec b$ means that a is strictly preferred to b. A strict order is defined from a preorder as $a \prec b$ if and only if $a \preceq b$ holds but $b \preceq a$ does not hold.

The equivalence, denoted by $\approx$, is defined as $a \approx b$ if and only if $a \preceq b$ and $b \preceq a$. Moreover, we define the incomparability, denoted by $\sim$, as $a \sim b$ if and only if neither $a \preceq b$ nor $b \preceq a$ holds. The set of minimal elements of A with respect to $\prec$, denoted by $Min(A, \prec)$, is defined as:

$$Min(A, \prec) = \{a \in A, \nexists b \in A : b \prec a\}$$

A total preorder $\leq$ on a finite set A is a reflexive and transitive binary relation such that $\forall a, b \in A$, either $a \leq b$ or $b \leq a$ holds.

A Refresher on Standard Lexicographic Inference

A totally preordered belief base $(\Sigma, \leq)$ is a set Σ of classical formulae equipped with a total preorder $\leq$ reflecting the priority relation that exists between its formulae. $(\Sigma, \leq)$ can be viewed as a stratified belief base $\Sigma = S_1 \cup \cdots \cup S_m$ such that the formulae in S_i have the same level of priority and have a higher priority than the ones in S_j with $j > i$.

Many coherence-based approaches have been developed to reason from totally preordered belief bases. Following Pinkas and Loui's analysis (Pinkas and Loui 1992), coherence-based inference can be considered as a two step process consisting first in generating some preferred consistent subbases and then using classical inference from some of them. Examples of coherence-based approaches are the possibilistic inference (Dubois, Lang, and Prade 1994), the inclusion-preference inference (Brewka 1989) and the lexicographic inference (Benferhat et al. 1993; Lehmann 1995) which is the focus of this paper.

The lexicographic preference between consistent subbases of a totally preordered belief base is defined as follows:

Definition 1 *Given a totally preordered belief base $(\Sigma, \leq) = S_1 \cup \ldots \cup S_m$, let A and B be two consistent subbases of Σ. Then,*

- *$A <_{lex} B$ iff $\exists i$, $1 \leq i \leq m$ such that $|S_i \cap A| > |S_i \cap B|$ [1] and $\forall j$, $j < i$, $|S_j \cap B| = |S_j \cap A|$*
- *$A =_{lex} B$ iff $\forall i$, $1 \leq i \leq m$ such that $|S_i \cap A| = |S_i \cap B|$.*

Let $Lex(\Sigma, \leq)$ denote the set of all the lexicographically preferred consistent subbases of Σ, namely

$$Lex(\Sigma, \leq) = Min(CONS(\Sigma), <_{lex}).$$

It is easy to check that every element of $Lex(\Sigma, \leq)$ is a maximal (with respect to set inclusion) consistent subbase of Σ.

Then, the lexicographic inference from $(\Sigma, \leq)$ is defined by:

Definition 2 *Let $(\Sigma, \leq)$ be a totally preordered belief base and let ψ be a propositional formula. ψ is a lexicographic consequence of Σ, denoted by $\Sigma \vdash_{lex} \psi$, iff ψ is a classical consequence of any lexicographically preferred consistent subbase of Σ, namely*

$$(\Sigma, \leq) \vdash_{lex} \psi \text{ iff } \forall B \in Lex(\Sigma, \leq) : B \vdash \psi.$$

Now, the inclusion-preference inference, denoted by $\vdash_{incl}$, is defined in the same manner as the lexicographic inference by substituting the cardinality comparison by the set inclusion operator, namely

$$A <_{incl} B \text{ iff } \exists i, 1 \leq i \leq m \text{ such that } (S_i \cap B) \subset (S_i \cap A) \text{ and } \forall j,\ j < i,\ (S_j \cap B) = (S_j \cap A).$$

[1] $|A|$ denotes the number of formulae of A.

As to the possibilistic inference which is denoted by $\vdash_{\pi}$, it comes down to classical inference from the classical base $\{\bigcup_{i=1}^{s-1} S_i\}$, where s is the smallest index such that $\bigcup_{i=1}^{s} S_i$ is inconsistent.

A Lexicographic Inference Based on Compatible Totally Preordered Bases

We first recall the notion of totally preordered belief bases compatible with a given partially preordered belief base. Intuitively, a totally preordered belief base $(\Sigma, \leq)$ is said to be compatible with a partially preordered belief base $(\Sigma, \preceq)$ if and only if the total preorder $\leq$ extends the partial preorder $\preceq$. Namely, it preserves strict and non-strict preference relations between every two formulae. More formally:

Definition 3 *Let $(\Sigma, \preceq)$ be a partially preordered belief base and let $\prec$ be the strict order associated with $\preceq$. A totally preordered belief base $(\Sigma, \leq)$ is said to be compatible with $(\Sigma, \preceq)$ if and only if:*

- $\forall \varphi, \phi \in \Sigma$: *if* $\varphi \preceq \phi$ *then* $\varphi \leq \phi$,
- $\forall \varphi, \phi \in \Sigma$: *if* $\varphi \prec \phi$ *then* $\varphi < \phi$.

We denote by $\mathcal{C}(\Sigma, \preceq)$ the set of all totally preordered belief bases compatible with $(\Sigma, \preceq)$.

Now, we define a lexicographic inference for partially preordered belief bases that relies on this notion of compatible totally preordered belief bases as follows:

Definition 4 *Let $(\Sigma, \preceq)$ be a partially preordered belief base and let ψ be a propositional formula. Then ψ is a $\mathcal{C}$-lexicographic conclusion of $(\Sigma, \preceq)$, denoted by $(\Sigma, \preceq) \vdash_{lex}^{\mathcal{C}} \psi$, if and only if ψ is a classical lexicographic inference of every totally preordered belief base compatible with $(\Sigma, \preceq)$, namely*

$$(\Sigma, \preceq) \vdash_{lex}^{\mathcal{C}} \psi \text{ iff } \forall (\Sigma, \leq) \in \mathcal{C}(\Sigma, \preceq)\text{: } (\Sigma, \leq) \vdash_{lex} \psi.$$

We note that this inference relation is very natural. Indeed, the notion of compatible totally preordered bases is very intuitive and has been considered in other previous works (but not with respect to lexicographic inference) like for example the possibilistic inference extension given in (Benferhat, Lagrue, and Papini 2003) and the inclusion-preference inference extension provided in (Junker and Brewka 1989).

Note that in Definition 4, if one replaces $\vdash_{lex}^{\mathcal{C}}$ and $\vdash_{lex}$ by respectively $\vdash_{\pi}^{\mathcal{C}}, \vdash_{\pi}$ and $\vdash_{incl}^{\mathcal{C}}, \vdash_{incl}$ then we get $(\Sigma, \preceq) \vdash_{\pi}^{\mathcal{C}} \psi$ implies $(\Sigma, \preceq) \vdash_{inlc}^{\mathcal{C}} \psi$ which itself implies $(\Sigma, \preceq) \vdash_{lex}^{\mathcal{C}} \psi$, and the converses are false. The proof is immediate, since with totally preordered knowledge bases, it has been shown in (Benferhat et al. 1993) that each possibilistic conclusion is an inclusion-preference conclusion and also a lexicographic conclusion.

Moreover, in Definition 4 we consider the whole set of compatible totally ordered preferred belief bases. One may just consider one compatible totally ordered belief base. This can be done by using some information measures or by selecting the most compact one. This way makes sense in some applications. However in this paper, we prefer to avoid such a choice, since this may lead to some risky or adventurous consequence relations.

Let us now illustrate the $\mathcal{C}$-lexicographic inference relation with the following example which will be used in the whole paper:

Example 1 *This example is inspired from an application within the framework of the European VENUS project regarding managing archaeological information. One of the tasks is the measurement using archaeological knowledge and requires the fusion of information coming from different sources.*

For example, several agents equipped with different measuring tools observe an amphora on an underwater site. According to the first agent, this amphora is a Dressel 7 (d) and the height of the amphora is 70 cm (h). According to the second one, the amphora has a rim diameter of 15 cm (r) and a height of 75 cm ($\neg h$). According to the third one, the amphora has a rim diameter of 18 cm ($\neg r$). The first agent is less reliable than the others. We do not know if the second agent is more reliable than the third. Another source is archaeological information on Dressel 7 amphora which has been copied in an XML file by a fourth agent who is less reliable than the three others. According to the XML file, a Dressel 7 has a rim diameter between 9.00 and 15.00 cm and a height between 50.00 and 70.00 cm ($d \rightarrow r \wedge h$).

The set of formulae is $\Sigma = \{d \rightarrow r \wedge h, d, r, \neg h, \neg r, h\}$ and the partial preorder $\preceq$ on Σ is given in Figure 1, where the arc "$a \rightarrow b$" means that $b \prec a$.

$$\begin{tikzcd}
\neg r & & r \approx \neg h \\
& d \approx h \arrow[ul] \arrow[ur] & \\
& d \rightarrow r \wedge h \arrow[u] &
\end{tikzcd}$$

Figure 1: The partial preorder $\preceq$ on Σ

There are three total preorders $\{\leq^1, \leq^2, \leq^3\}$ which are compatible with the partial preorder $\preceq$, i.e., $\mathcal{C}(\Sigma, \preceq) = \{(\Sigma, \leq^1), (\Sigma, \leq^2), (\Sigma, \leq^3)\}$ given in Figure 2 and Figure 3. For lack of space, we only provide the relation between maximally consistent subbases and not between all consistent subbases. This has no incidence on the definition of $\mathcal{C}$-lexicographic consequence relation. The set of maximal consistent subbases of Σ is $MCONS(\Sigma) = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ with $A_1 = \{d \rightarrow r \wedge h, d, r, h\}$, $A_2 = \{d \rightarrow r \wedge h, \neg r, h\}$, $A_3 = \{d \rightarrow r \wedge h, \neg h, \neg r\}$, $A_4 = \{d \rightarrow r \wedge h, r, \neg h\}$, $A_5 = \{d, \neg r, h\}$, $A_6 = \{d, \neg h, \neg r\}$ and $A_7 = \{d, r, \neg h\}$.

The lexicographic preferences $\leq_{lex}^1, \leq_{lex}^2$ and $\leq_{lex}^3$ on the maximal $MCONS(\Sigma)$ with respect to the compatible belief bases $(\Sigma, \leq^1), (\Sigma, \leq^2)$ and $(\Sigma, \leq^3)$ respectively are given in Figure 2 and Figure 3.

Finally, we deduce that

$$
\begin{array}{cc}
(\Sigma, \leq^1) & \leq^1_{lex} \\
\neg r =^1 r =^1 \neg h & A_6 =^1_{lex} A_7 \\
\uparrow & \uparrow \\
d =^1 h & A_3 =^1_{lex} A_4 \\
\uparrow & \uparrow \\
d \to r \wedge h & A_1 \\
 & \uparrow \\
 & A_5 \\
 & \uparrow \\
 & A_2
\end{array}
$$

Figure 2: The total preorder $\leq^1_{lex}$ on $MCONS(\Sigma)$

$$
\begin{array}{cc|cc}
(\Sigma, \leq^2) & \leq^2_{lex} & (\Sigma, \leq^3) & \leq^3_{lex} \\
\neg r & A_6 & r =^3 \neg h & A_7 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
r =^2 \neg h & A_3 & \neg r & A_4 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
d =^2 h & A_5 & d =^3 h & A_6 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
d \to r \wedge h & A_2 & d \to r \wedge h & A_3 \\
 & \uparrow & & \uparrow \\
 & A_7 & & A_1 \\
 & \uparrow & & \uparrow \\
 & A_4 & & A_5 \\
 & \uparrow & & \uparrow \\
 & A_1 & & A_2
\end{array}
$$

Figure 3: The total preorders $\leq^2_{lex}$ and $\leq^3_{lex}$ on $MCONS(\Sigma)$

$$\bigcup_{(\Sigma,\leq^i)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq^i) = \{A_6, A_7\} \cup \{A_6\} \cup \{A_7\} =$$
$\{A_6, A_7\}$*, from which for instance* d *is a* $\mathcal{C}$*-lexicographic consequence of* Σ*, since* $A_6 \vdash d$ *and* $A_7 \vdash d$*.*

A New Preference Between Consistent Subbases

The aim of this section is to provide the counterpart of Definition 1 for partially preordered belief bases. Clearly, the main difficulty is how to define the concept of "cardinality" (namely how to use the idea of counting) when pieces of information are only partially preordered. Let $(\Sigma, \preceq)$ be a partially preordered belief base. We first propose a partition of Σ, $\Sigma = E_1 \cup \ldots \cup E_n$ $(n \geq 1)$ as follows:

- $\forall i$, $1 \leq i \leq n$, we have $\forall \varphi, \varphi' \in E_i$: $\varphi \approx \varphi'$,
- $\forall i$, $1 \leq i \leq n$, $\forall j$, $1 \leq j \leq n$ with $i \neq j$, we have $\forall \varphi \in E_i$, $\forall \varphi' \in E_j$: $\varphi \not\approx \varphi'$.

In other words, each subset E_i represents an equivalence class of Σ with respect to $\approx$ which is an equivalence relation.

Then, we define a preference relation between the equivalence classes E_i's, denoted by $\prec_s$, as follows:

Definition 5 *Let* E_i *and* E_j *be two equivalence classes of* Σ *with respect to* $\approx$*. Then,*

$$E_i \prec_s E_j \textit{ iff } \exists \varphi \in E_i, \exists \varphi' \in E_j \textit{ such that } \varphi \prec \varphi'.$$

The relation $\prec_s$ can be equivalently given by:

$$E_i \prec_s E_j \text{ iff } \forall \varphi \in E_i, \forall \varphi' \in E_j : \varphi \prec \varphi'.$$

Besides, one can easily see that the preference relation $\prec_s$ over the equivalence classes E_i's is a partial strict order. In addition, $E_i \sim_s E_j$ if and only if neither $E_i \prec_s E_j$ nor $E_j \prec_s E_i$ holds.

This partition can be viewed as a generalization of the idea of stratification defined for totally preordered belief bases.

The following example illustrates this partition.

Example 2 *Let us consider again the partially preordered belief base* $(\Sigma, \preceq)$ *given in Example* 1*.* Σ *can be partitioned as follows:* $\Sigma = E_1 \cup E_2 \cup E_3 \cup E_4$ *with* $E_1 = \{\neg r\}$*,* $E_2 = \{r, \neg h\}$*,* $E_3 = \{d, h\}$ *and* $E_4 = \{d \to r \wedge h\}$*.*

According to Definition 5, we have : $E_1 \sim_s E_2$*,* $E_1 \prec_s E_3$*,* $E_1 \prec_s E_4$*,* $E_2 \prec_s E_3$*,* $E_2 \prec_s E_4$ *and* $E_3 \prec_s E_4$*. The partial strict order* $\prec_s$ *is illustrated in Figure 4.*

$$
\begin{array}{ccc}
E_1 & & E_2 \\
\nwarrow & & \nearrow \\
 & E_3 & \\
 & \uparrow & \\
 & E_4 &
\end{array}
$$

Figure 4: The partial strict order $\prec_s$ on Σ

We now provide a lexicographic preference relation between the consistent subbases of a partially preordered belief base $(\Sigma, \preceq)$, denoted by $\preceq_\Delta$, as follows:

Definition 6 *Let* $(\Sigma, \preceq)$ *be a partially preordered belief base and let* A *and* B *be two consistent subbases of* Σ*. Then,* A *is said to be lexicographically preferred to* B*, denoted by* $A \preceq_\Delta B$*, if and only if*

$$\begin{gathered}\forall i,\ 1 \leq i \leq n : \\ \textit{if } |E_i \cap B| > |E_i \cap A| \textit{ then } \exists j,\ 1 \leq j \leq n \textit{ such that} \\ |E_j \cap A| > |E_j \cap B| \textit{ and } E_j \prec_s E_i.\end{gathered}$$

The following proposition gives some properties of the preference relation $\preceq_\Delta$:

Proposition 1 *Let* $(\Sigma, \preceq)$ *be a partially preordered belief base. Then,*

- $\preceq_\Delta$ *is a partial preorder on the set of consistent subbases of* Σ*.*
- $\preceq_\Delta$ *satisfies the monotony property, namely*

$$\forall A, B \subseteq \Sigma, \textit{ if } B \subseteq A \textit{ then } A \preceq_\Delta B.$$

As usual, the strict partial order associated with $\preceq_\Delta$, is defined by: A is strictly lexicographically preferred to B, denoted by $A \prec_\Delta B$, iff $A \preceq_\Delta B$ and $B \not\preceq_\Delta A$. As for the corresponding equality, it is given by: A is lexicographically equal to B, denoted by $A \approx_\Delta B$, iff $A \preceq_\Delta B$ and $B \preceq_\Delta A$. The following proposition gives an equivalent definition of $\approx_\Delta$:

Proposition 2 *Let $(\Sigma, \preceq)$ be a partially preordered belief base and let A and B be two consistent subbases of Σ. Then,*

$$A \approx_\Delta B \textit{ iff } \forall i, 1 \le i \le n : |E_i \cap B| = |E_i \cap A|.$$

Clearly, the proposed lexicographic preference between consistent subbases of a partially preordered belief base (Definition 6) recovers the classical lexicographic preference between consistent subbases of a totally preordered belief base (Definition 1) as it is shown by the following proposition:

Proposition 3 *Given a totally preordered belief base $(\Sigma, \le) = S_1 \cup \ldots \cup S_m$, let A and B be two consistent subbases of Σ. Then,*

1. *$A <_{lex} B$ if and only if $A \prec_\Delta B$,*
2. *$A =_{lex} B$ if and only if $A \approx_\Delta B$.*

The following example illustrates this lexicographic preference.

Example 3 *Let us continue Example 1 and 2. The partial preorder $\preceq_\Delta$ on $MCONS(\Sigma)$, obtained using Definition 6, is illustrated in Figure 5. For example,*

- *$A_3 \prec_\Delta A_1$ since we have $i = 3$ such that $|E_3 \cap A_1| > |E_3 \cap A_3|$ and $\exists j, j = 1$ such that $|E_1 \cap A_3| > |E_1 \cap A_1|$ and $E_1 <_s E_3$.*
- *$A_4 \prec_\Delta A_1$ since we have $i = 3$ such that $|E_3 \cap A_1| > |E_3 \cap A_4|$ and $\exists j, j = 2$ such that $|E_2 \cap A_4| > |E_2 \cap A_1|$ and $E_2 <_s E_3$.*
- *$A_3 \sim_\Delta A_4$ since*
  - *$A_3 \not\prec_\Delta A_4$ since we have $i = 2$ such that $|E_2 \cap A_4| > |E_2 \cap A_3|$ but $\nexists j$ such that $|E_j \cap A_3| > |E_j \cap A_4|$ and $E_j <_s E_2$.*
  - *$A_4 \not\prec_\Delta A_3$ since we have $i = 1$ such that $|E_1 \cap A_3| > |E_1 \cap A_4|$ but $\nexists j$ such that $|E_j \cap A_4| > |E_j \cap A_3|$ and $E_j <_s E_1$.*

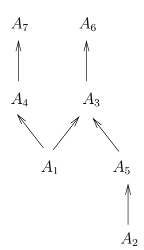

Figure 5: The partial preorder $\preceq_\Delta$ on $MCONS(\Sigma)$

Note that in Definition 6 if one uses inclusion set criteria instead of cardinality then Proposition 3 does not hold, namely the classical inclusion-preference inference will not be recovered.

A New Characterisation of Lexicographic Inference

Let $(\Sigma, \preceq)$ be a partially preordered belief base, $\mathfrak{L}ex(\Sigma, \preceq)$ denotes the set of all lexicographically preferred consistent subbases of $(\Sigma, \preceq)$, namely $\mathfrak{L}ex(\Sigma, \preceq) = Min(CONS(\Sigma), \prec_\Delta)$.

It can be checked that each element of $\mathfrak{L}ex(\Sigma, \preceq)$ is a maximal (with respect to set inclusion) consistent subbase of Σ since $\preceq_\Delta$ satisfies the monotony property (Proposition 1).

We are now able to provide a second lexicographic inference relation for partially preordered belief bases which is based on the preordering $\preceq_\Delta$ between consistent subbases:

Definition 7 *Let $(\Sigma, \preceq)$ be a partially preordered base and let ψ be a formula. ψ is a $\mathcal{P}$-lexicographic conclusion of $(\Sigma, \preceq)$, denoted by $(\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi$, iff ψ is a classical consequence of every preferred consistent subbase of $(\Sigma, \preceq)$ with respect to $\preceq_\Delta$, namely*

$$(\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi \textit{ iff } \forall B \in \mathfrak{L}ex(\Sigma, \preceq)\text{: } B \vdash \psi.$$

The following lemma, needed for the proof of one of our main result, says that if A is lexicographically preferred to B then A is lexicographically preferred to B in every compatible totally preordered base.

Lemma 1 *Let $(\Sigma, \preceq)$ be a partially preordered belief base and let A and B be two consistent subbases of Σ. Let $(\Sigma, \le)$ be a totally preordered belief base compatible with $(\Sigma, \preceq)$. Then,*

- *if $A \prec_\Delta B$ then $A <_{lex} B$.*
- *if $A \approx_\Delta B$ then $A =_{lex} B$.*

Interestingly enough, the lexicographic inference relation presented in this section is covered by the natural lexicographic inference given by Definition 4 as shown by the following proposition:

Proposition 4 *Let $(\Sigma, \preceq)$ be a partially preordered belief base and let ψ be a propositional formula. Then,*

$$\textit{if } (\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi \textit{ then } (\Sigma, \preceq) \vdash^{\mathcal{C}}_{lex} \psi.$$

Let us illustrate this lexicographic inference:

Example 4 *Continue Example 3. Clearly, we have $\mathfrak{L}ex(\Sigma, \preceq) = \{A_6, A_7\}$.*

Moreover, one can easily see that $\mathfrak{L}ex(\Sigma, \preceq) = \bigcup_{(\Sigma, \le^i) \in \mathcal{C}(\Sigma, \preceq)} Lex(\Sigma, \le^i)$ *where* $\bigcup_{(\Sigma, \le^i) \in \mathcal{C}(\Sigma, \preceq)} Lex(\Sigma, \le^i)$ *has already been computed in Example* 1.

Semantic Counterpart and Properties

This section briefly shows that the lexicographical inference relation applied to partially preordered belief bases can be defined at the semantic level. We also show that a notion of preferential entailment for our lexicographic inference, in the style of (Shoham 1988), can be naturally produced.

Roughly speaking, an interpretation model ω is preferred to an interpretation ω' if the set of formulae satisfied by ω is lexicographically preferred to the one satisfied by ω'. More formally,

Definition 8 *Let ω and ω' be two interpretations. ω is said to be lexicographically preferred to ω', denoted by $\omega \preceq^{\Delta}_{\mathcal{W}} \omega'$, iff $[\omega] \preceq_{\Delta} [\omega']$ where $[\omega]$ denotes the set of formulae of Σ satisfied by ω.*

Let $Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$ denote the set of preferred models of $\mathcal{W}$ with respect to $\prec^{\Delta}_{\mathcal{W}}$.

Then, we give a semantic inference relation defined as follows:

Definition 9 *Let $(\Sigma, \preceq)$ be a partially preordered belief base and let ψ be a propositional formula. ψ is a $\mathcal{S}$-lexicographic conclusion of $(\Sigma, \preceq)$, denoted by $(\Sigma, \preceq) \models^{\mathcal{S}}_{lex} \psi$, if and only if ψ is satisfied by every preferred interpretation, namely*
$(\Sigma, \preceq) \models^{\mathcal{S}}_{lex} \psi$ *iff* $\forall \omega \in Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}}), \omega \models \psi$.

and we show that:

Proposition 5 $(\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi$ *iff* $(\Sigma, \preceq) \models^{\mathcal{S}}_{lex} \psi$.

Example 5 *We illustrate our semantic approach with the example 1. Let $\mathcal{W}$ be the set of interpretations and $[\omega]$ the set of formulae of Σ satisfied by ω :*

$\mathcal{W}$	d	h	r	$[\omega]$	A_i
ω_0	$\neg d$	$\neg h$	$\neg r$	$\{d \to r \wedge h, \neg h, \neg r\}$	A_3
ω_1	$\neg d$	$\neg h$	r	$\{d \to r \wedge h, r, \neg h\}$	A_4
ω_2	$\neg d$	h	$\neg r$	$\{d \to r \wedge h, h, \neg r\}$	A_2
ω_3	$\neg d$	h	r	$\{d \to r \wedge h, h, r\}$	
ω_4	d	$\neg h$	$\neg r$	$\{d, \neg h, \neg r\}$	A_6
ω_5	d	$\neg h$	r	$\{d, \neg h, r\}$	A_7
ω_6	d	h	$\neg r$	$\{d, \neg r, h\}$	A_5
ω_7	d	h	r	$\{d \to r \wedge h, d, r, h\}$	A_1

Reusing Example 3 where we already computed preferences between A_i, the ordering between interpretations is illustrated in figure 6.

We can easily check:

$$\bigcup_{B \in \mathfrak{Lex}(\Sigma, \preceq)} Mod(B) = Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$$

Indeed, $\mathfrak{Lex}(\Sigma, \preceq) = \{A_6, A_7\}$ and since $A_6 = \{d, \neg h, \neg r\}$ and $A_7 = \{d, r, \neg h\}$, $Mod(A_6) = \{\omega_4\}$ and $Mod(A_7) = \{\omega_5\}$. Thus,

$$\bigcup_{B \in \mathfrak{Lex}(\Sigma, \preceq)} Mod(B) = \{\omega_4, \omega_5\} = Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$$

By consequence, $(\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi$ iff $(\Sigma, \preceq) \models^{\mathcal{S}}_{lex} \psi$.

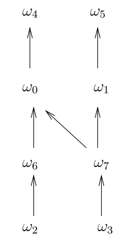

Figure 6: The partial preorder $\preceq^{\Delta}_{\mathcal{W}}$ on $\mathcal{W}$

We now analyse the non-monotonic properties of our inference relation. First, we need to extend our inference relation such that it can be defined between two formulae with respect to a partially preordered belief base:

Definition 10 *Let $(\Sigma, \preceq)$ be a belief base and ϕ, ψ be two formulae. Let $(\Sigma', \preceq')$ be a new partially preordered base where $\Sigma' = \Sigma \cup \{\phi\}$ and $\preceq'$ is such that:*

- $\forall \alpha, \beta \in \Sigma, \alpha \preceq \beta$ *iff* $\alpha \preceq' \beta$
- $\forall \alpha \in \Sigma, \phi \preceq' \alpha$, *namely ϕ is the most important formula in Σ'.*

Then a formula ψ is said to be a Lex-consequence of ϕ with respect to the belief base Σ, denoted by $\phi \mid\!\sim_{lex} \psi$, iff $(\Sigma', \preceq') \vdash^{\mathcal{P}}_{lex} \psi$.

The following result generalizes Proposition 5.

Proposition 6 *Let ϕ and ψ be two formulae.*
$\phi \mid\!\sim_{lex} \psi$ *iff* $\forall \omega \in Min(Mod(\phi), \prec^{\Delta}_{\mathcal{W}}), \omega \models \psi$.

The proof follows from Proposition 5 where ϕ is preferred to every formula of $(\Sigma, \preceq)$.

When the belief base is totally preordered, then it is well-known that the inference $\mid\!\sim_{lex}$ satisfies the rules of System P and rational monotony. For partially preordered belief bases, we have:

Proposition 7 *The inference $\mid\!\sim_{lex}$ satisfies the rational postulates of System P but fails to satisfy rational monotony.*

For the System P, the proof can be shown by exploiting a representation theorem between preferential entailments satisfying System P and its semantics characterisation based on a partial preorder over interpretation (Kraus, Lehmann, and Magidor 1990). For rational monotony, the following provides the counter-example:

Counter-example 1 *Let us recall that Rational Montony is defined by : If $\phi \mid\!\sim \psi$, and $\phi \not\mid\!\sim \neg\delta$ then $\phi \wedge \delta \mid\!\sim \psi$. Let $\Sigma = \{p, q, \neg r\}$ such that $p \prec \neg r$, $q \prec \neg r$ and $p \sim q$. $E_1 = q$, $E_2 = p$ and $E_3 = \neg r$. The set of formulae of Σ satisfied by ω is :*

$\mathcal{W}$	p	q	r	*equivalent* $[\omega]$
ω_0	$\neg p$	$\neg q$	$\neg r$	$\{\neg r\}$
ω_1	$\neg p$	$\neg q$	r	$\emptyset$
ω_2	$\neg p$	q	$\neg r$	$\{q, \neg r\}$
ω_3	$\neg p$	q	r	$\{q\}$
ω_4	p	$\neg q$	$\neg r$	$\{p, \neg r\}$
ω_5	p	$\neg q$	r	$\{p\}$
ω_6	p	q	$\neg r$	$\{p, q, \neg r\}$
ω_7	p	q	r	$\{p, q\}$

The preorder $\preceq^{\triangle}_{\mathcal{W}}$ on $\mathcal{W}$ is illustrated figure 7.

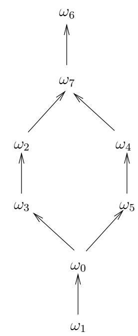

Figure 7: The preorder $\prec^{\triangle}_{\mathcal{W}}$ on $\mathcal{W}$

Let ϕ, ψ and δ three formulae such that $\phi = (\neg p \vee \neg q) \wedge (\neg p \vee r)$, $\psi = p \vee \neg r$ and $\delta = r$. We have $Mod(\phi) = \{\omega_0, \omega_1, \omega_2, \omega_3, \omega_5\}$ and $Min(Mod(\phi), \preceq^{\triangle}_{\mathcal{W}}) = \{\omega_2, \omega_5\}$. We checked that $\phi \mid\!\sim_{lex} \psi$ since $Mod(\psi) = \{\omega_0, \omega_2, \omega_4, \omega_5, \omega_6, \omega_7\}$, $\omega_2 \models \psi$ and $\omega_5 \models \psi$. Likewise we do not have $\phi \mid\!\sim_{lex} \neg\delta$ since $Mod(\neg\delta) = \{\omega_0, \omega_2, \omega_4, \omega_6\}$, $\omega_2 \models \neg\delta$ and $\omega_5 \not\models \neg\delta$. On the other hand, $Mod(\delta) = \{\omega_1, \omega_3, \omega_5, \omega_7\}$, $Mod(\phi \wedge \delta) = \{\omega_1, \omega_3, \omega_5\}$ and $Min(Mod(\phi \wedge \delta), \preceq^{\triangle}_{\mathcal{W}}) = \{\omega_3, \omega_5\}$ but we do not have $(\phi \wedge \delta) \mid\!\sim_{lex} \psi$ since $\omega_3 \not\models \psi$ and $\omega_5 \models \psi$. Therefore, the inference $\mid\!\sim_{lex}$ does not satisfy the rational monotony.

Conclusion

This paper combines the advantages of both lexicographic inference and partially preordered belief bases for the sake of reasoning under inconsistency. It proposes a lexicographic inference for partially preordered belief bases that extends the classical lexicographic one.

More precisely, the paper provides two definitions. The first one consists in applying the classical lexicographic inference from all the compatible totally preordered belief bases. As to the second one, it stems from the construction of a new cardinality-based partial preorder on consistent subbases. It then applies classical inference on preferred subbases according to this partial preorder. It turns out that the second inference is covered by the first one.

This work calls for several perspectives. The major is its application to alarm correlation in intrusion detection and to archaeological excavation.

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References

Benferhat, S.; Dubois, D.; Cayrol, C.; Lang, J.; and Prade, H. 1993. Inconsistency management and prioritized syntax-based entailment. In *IJCAI'03*, 640–647.

Benferhat, S.; Bonnefon, J.-F.; and Da Silva Neves, R. 2004. An experimental analysis of possibilistic default reasoning. In *KR'04*, 130–140.

Benferhat, S.; Lagrue, S.; and Papini, O. 2003. A possibilistic handling of partially ordered information. In *UAI'03*, 29–36.

Benferhat, S.; Lagrue, S.; and Papini, O. 2004. Reasoning with partially ordered information in a possibilistic framework. *Fuzzy Sets and Systems* 144:25–41.

Brewka, G. 1989. Preferred sutheories: an extende logical framework for default reasoning. In *IJCAI'89*, 1043–1048.

da Costa, N. C. A. 1974. Theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic.* 15:497–510.

Dubois, D.; Lang, J.; and Prade, H. 1994. Possibilistic logic. *Handbook of Logic in Articial Intelligence and Logic Programming.* 3:439–513.

Junker, U., and Brewka, G. 1989. Handling partially ordered defaults in TMS. In *IJCAI'89*, 1043–1048.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence.* 44(1-2):167–207.

Lehmann, D. J. 1995. Another perspective on default reasoning. *Annals of Mathematics and Artificial Intelligence* 15(1):61–82.

Nebel, B. 1994. Base revision operations and schemes: semantics, representation and complexity. In *ECAI'94*, 341–345.

Pinkas, G., and Loui, R. P. 1992. Reasoning from inconsistency: A taxonomy of principles for resolving conflict. In *KR'92*, 709–719.

Resher, N., and Manor, R. 1970. On inference from inconsistent premises. *Theory and Decision.* 1:179–219.

Ritterskamp, M., and Kern-Isberner, G. 2008. Preference-Based Default Reasoning. In *FLAIRS'08*.

Shoham, Y. 1988. *Reasoning about change: Time and Causation from the standpoint of Artificial Intelligence.* MIT press.

Williams, M. 1995. Iterated theory base change: a computational model. In *IJCAI'95*, 1541–1547.

Appendix

Proposition 1
Let $(\Sigma, \preceq)$ be a partially preordered belief base. Then,

- *$\preceq_\Delta$ is a partial preorder on the set of consistent subbases of Σ.*
- *$\preceq_\Delta$ satisfies the monotony property, namely*

$$\forall A, B \subseteq \Sigma, \text{ if } B \subseteq A \text{ then } A \preceq_\Delta B.$$

Proof.

- $\preceq_\Delta$ is a partial preorder over $CONS(\Sigma)$:
  - **$\preceq_\Delta$ is reflexive:**
    Let $A \in CONS(\Sigma)$.
    Then, obviously we have $\forall i, 1 \leq i \leq n : |E_i \cap A| = |E_i \cap A|$.
    So, $A \preceq_\Delta A$ which means that $\preceq_\Delta$ is a reflexive relation.
  - **$\preceq_\Delta$ is not antisymmetric:**
    Let us consider the following counter example:
    $\Sigma = \{\varphi, \phi, \psi, \chi\}$ with $\varphi \approx \chi$ and $\phi \approx \psi$.
    Let $A, B \in CONS(\Sigma)$ such that $A = \{\varphi, \phi\}$ and $B = \{\psi, \chi\}$.
    One can easily see that $A \preceq_\Delta B$ and $B \preceq_\Delta A$ but A is not identical to B.
    Hence, $\preceq_\Delta$ is not an antisymmetric relation.
  - **$\preceq_\Delta$ is transitive:**
    Let $A, B, C \in CONS(\Sigma)$ and assume that $A \preceq_\Delta B$ and $B \preceq_\Delta C$ and let us show that $A \preceq_\Delta C$.
    Assume that $A \preceq_\Delta C$ does not hold. Namely, there exists i, $1 \leq i \leq n$ such that $|E_i \cap C| > |E_i \cap A|$ and $\nexists j, 1 \leq j \leq n$ such that $E_j \prec_s E_i$ and $|E_j \cap A| > |E_i \cap C|$ ... **(Hypothesis 1)**
    In fact, we can distinguish two cases:
    * **Case 1:** $|E_i \cap B| > |E_i \cap A|$:
      Let $F = \{E_j, 1 \leq j \leq n : |E_j \cap A| > |E_j \cap B|$ and $E_j \prec_s E_i\}$.
      Since $A \preceq_\Delta B$ then $F \neq \emptyset$.
      In particular, let $E_j \in Min(F, \prec_s)$.
      Now, according to Hypothesis 1, $|E_j \cap C| \geq |E_j \cap A|$ thus $|E_j \cap C| > |E_j \cap B|$.
      Then, given that $B \preceq_\Delta C$, let $E_k, 1 \leq k \leq n$ be such that $E_k \prec_s E_j$ and $|E_k \cap B| > |E_k \cap C|$.
      Once again, according to Hypothesis 1, $|E_k \cap C| \geq |E_k \cap A|$ so $|E_k \cap B| > |E_k \cap A|$.
      Since $A \preceq_\Delta B$, there must exist $l, 1 \leq l \leq n$ such that $E_l \prec_s E_k$ and $|E_l \cap A| > |E_l \cap B|$.
      Clearly, we have $E_l \prec_s E_j$ and $E_l \prec_s E_i$ (by transitivity of $\prec_s$).
      Hence, we deduce that $E_l \in F$ and $E_l \prec_s E_j$ but this is incoherent with the fact that $E_j \in Min(F, \prec_s)$.
    * **Case 2:** $|E_i \cap B| \leq |E_i \cap A|$:
      Obviously, $|E_i \cap B| < |E_i \cap C|$.
      Let $G = \{E_j, 1 \leq j \leq n : |E_j \cap B| > |E_j \cap C|$ and $E_j \prec_s E_i\}$.
      Since $B \preceq_\Delta C$, $G \neq \emptyset$.
      In particular, let $E_j \in Min(G, \prec_s)$.
      Now, according to Hypothesis 1, $|E_j \cap C| \geq |E_j \cap A|$ thus $|E_j \cap B| > |E_j \cap A|$.
      Then, given that $A \preceq_\Delta B$, let $E_k, 1 \leq k \leq n$ be such that $E_k \prec_s E_j$ and $|E_k \cap A| > |E_k \cap B|$.
      Once again, according to Hypothesis 1, $|E_k \cap C| \geq |E_k \cap A|$ so $|E_k \cap C| > |E_k \cap B|$.
      Since $B \preceq_\Delta C$, there must exist $l, 1 \leq l \leq n$ such that $E_l \prec_s E_k$ and $|E_l \cap B| > |E_l \cap C|$.
      Clearly, we have $E_l \prec_s E_j$ and $E_l \prec_s E_i$ (by transitivity of $\prec_s$).
      Hence, we deduce that $E_l \in G$ and $E_l \prec_s E_j$ but this is incoherent with the fact that $E_j \in Min(G, \prec_s)$.

    Therefore, $\forall i$, $1 \leq i \leq n$: if $|E_i \cap C| > |E_i \cap A|$ then $\exists j, 1 \leq j \leq n$ such that $E_j \prec_s E_i$ and $|E_j \cap A| > |E_i \cap C|$.
    This means that $A \preceq_\Delta C$.
- $\preceq_\Delta$ satisfies the **monotony** property:
  Let $A, B \in CONS(\Sigma)$. $B \subseteq A$ implies that $\forall i$, $1 \leq i \leq n$: $|E_i \cap A| \geq |E_i \cap B|$.
  Thus, according to Definition 6, $A \preceq_\Delta B$. ■

Proposition 2
Let $(\Sigma, \preceq)$ be a partially preordered belief base and let A and B be two consistent subbases of Σ. Then,

$$A \approx_\Delta B \textit{ iff } \forall i, 1 \leq i \leq n : |E_i \cap B| = |E_i \cap A|.$$

Proof.

- By definition of $\preceq_\Delta$, we have $\forall i, 1 \leq i \leq n : |E_i \cap A| = |E_i \cap B|$ implies that $A \preceq_\Delta B$ and $B \preceq_\Delta A$, i.e., $A \approx_\Delta B$.
- Now let us show that $A \approx_\Delta B$ implies that $\forall i, 1 \leq i \leq n : |E_i \cap A| = |E_i \cap B|$.
  In fact, let us suppose that $A \approx_\Delta B$ and $\exists i, 1 \leq i \leq n : |E_i \cap A| \neq |E_i \cap B|$. We can distinguish two cases:
  - **Case 1:** $A \approx_\Delta B$ and $\exists i, 1 \leq i \leq n$ such that $|E_i \cap B| > |E_i \cap A|$:
    Let $F = \{E_i, 1 \leq i \leq n$ such that $|E_i \cap B| > |E_i \cap A|\}$. $F \neq \emptyset$ so let $E_i \in Min(F, \prec_s)$.
    Since $A \preceq_\Delta B$, $\exists j, 1 \leq j \leq n$ such that $E_j \prec_s E_i$ and $|E_j \cap A| > |E_j \cap B|$.
    Besides, since $B \preceq_\Delta A$, $\exists k$ such that $E_k \prec_s E_j$ and $|E_k \cap B| > |E_k \cap A|$.
    Clearly enough, $E_k \in F$ and $E_k \prec_s E_i$ (by transitivity of $\prec_s$) but this is incoherent with the fact that $E_i \in Min(F, \prec_s)$.
  - **Case 2:** $A \approx_\Delta B$ and $|E_i \cap A| > |E_i \cap B|$: Similar proof. ■

Proposition 3
Given a totally preordered belief base $(\Sigma, \leq) = S_1 \cup \ldots \cup S_m$, let A and B be two consistent subbases of Σ. Then,

1. $A <_{lex} B$ *if and only if* $A \prec_{\Delta} B$,
2. $A =_{lex} B$ *if and only if* $A \approx_{\Delta} B$.

Proof.

Note that in the case of a totally ordered belief base $(\Sigma, \leq) = (S_1, \ldots, S_m)$, the equivalence classes E_i's are nothing than the strata S_i's and $S_i \prec_s S_j$ iff $i < j$.

1. $A <_{lex} B$ if and only if $A \prec_{\Delta} B$:
   - Let us show that $A <_{lex} B$ implies $A \prec_{\Delta} B$:
     Let $A <_{lex} B$. By definition, $\exists i$, $1 \leq i \leq m$ such that $|S_i \cap A| > |S_i \cap B|$ and $\forall j$, $j < i$, $|S_j \cap B| = |S_j \cap A|$.
     Then, $\forall k, 1 \leq k \leq m$, if $|S_k \cap B| > |S_k \cap A|$ then $k > i$, i.e., $\exists i$ such that $|S_i \cap A| > |S_i \cap B|$ with $i < k$.
     This means that $A \preceq_{\Delta} B$.
     Besides, $\exists i, 1 \leq i \leq m$ such that $|S_i \cap A| > |S_i \cap B|$ and $\nexists j, 1 \leq j \leq m$ such that $|S_j \cap B| > |S_j \cap A|$ and $j < i$ which means that $B \npreceq_{\Delta} A$.
     Hence, $A \prec_{\Delta} B$.
   - Let us show that $A \prec_{\Delta} B$ implies $A <_{lex} B$:
     Let $A \prec_{\Delta} B$.
     Then, $\exists i, 1 \leq i \leq m$ such that $|S_i \cap A| > |S_i \cap B|$ and $\forall j, 1 \leq j \leq m$, we have $|S_j \cap B| \leq |S_j \cap A|$ since $B \not\prec_{\Delta} A$.
     Clearly, this means that $A <_{lex} B$.
2. $A <_{lex} B$ if and only if $A \approx_{\Delta} B$:
   This directly follows from Definition 1 and Proposition 2. ∎

Lemma 1
Let $(\Sigma, \preceq)$ be a partially preordered belief base and let A and B be two consistent subbases of Σ. Let $(\Sigma, \leq)$ be a totally preordered belief base compatible with $(\Sigma, \preceq)$. Then,

- *if $A \prec_{\Delta} B$ then $A <_{lex} B$.*
- *if $A \approx_{\Delta} B$ then $A =_{lex} B$.*

Proof.

Let $(\Sigma, \leq) = S_1 \cup \ldots \cup S_m$ be a totally ordered belief base compatible with $(\Sigma, \preceq)$.

1. $A \prec_{\Delta} B$ implies $A <_{lex} B$:
   We recall that $A \prec_{\Delta} B$ iff $A \preceq_{\Delta} B$ and $B \npreceq_{\Delta} A$. We can distinguish two cases:
   - **Case 1:** $\nexists i, 1 \leq i \leq n$ such that $|E_i \cap B| > |E_i \cap A|$. i.e., $\forall i$, $1 \leq i \leq n$, $|E_i \cap B| \leq |E_i \cap A|$.
     First, note that $\forall j$, $1 \leq j \leq m$, we have $S_j = \bigcup_{i \in X} E_i$ where $X \subseteq \{1, \ldots, n\}$.
     So, $|S_j \cap B| = |(\bigcup_{i \in X} E_i) \cap B| = |\bigcup_{i \in X}(E_i \cap B)| = \sum_{i \in X} |E_i \cap B|$ since E_i's are disjoint by definition.
     Similarly, we obtain $|S_j \cap A| = \sum_{i \in X} |E_i \cap A|$.
     Then, according to the hypothesis of this case, we conclude that $\sum_{i \in X} |E_i \cap B| \leq \sum_{i \in X} |E_i \cap A|$, i.e., $|S_j \cap B| \leq |S_j \cap A|$.
     On the other hand, $B \npreceq_{\Delta} A$, i.e., $\exists k, 1 \leq k \leq n$ such that $|E_k \cap A| > |E_k \cap B|$ and $\nexists l, 1 \leq l \leq n$ such that $E_l \prec_s E_k$ and $|E_l \cap B| > |E_l \cap A|$.
     Now, let S_i $(1 \leq i \leq m)$ be a stratum containing a class E_k checking the previous property, i.e., $S_i = E_k \cup \bigcup_{j \in Y} E_j$ where $Y \subseteq \{1, \ldots, n\}$.
     So, $|S_i \cap B| = |(\bigcup_{j \in Y} E_j \cup E_k) \cap B| = |\bigcup_{j \in Y}(E_j \cap B) \cup (E_k \cap B)| = \sum_{j \in Y} |E_j \cap B| + |E_k \cap B|$ since E_j's are disjoint by definition.
     Similarly, we derive that $|S_i \cap A| = \sum_{j \in Y} |E_j \cap A| + |E_k \cap A|$.
     Clearly, $|S_i \cap B| < |S_i \cap A|$.
     So, from one hand we have $\forall j, 1 \leq j \leq m : |S_j \cap B| \leq |S_j \cap A|$ and from the other hand $\exists i, 1 \leq i \leq m : |S_i \cap B| < |S_i \cap A|$.
     Consequently, $A <_{lex} B$.
   - **Case 2:** $\exists k, 1 \leq k \leq n$ such that $|E_k \cap B| > |E_k \cap A|$.
     Let S_i be the first stratum (with the smallest i) containing a class E_k that checks the previous property.
     Since $A \prec_{\Delta} B$, there must exist a class E_p such that $|E_p \cap A| > |E_p \cap B|$ and $E_p \prec_s E_k$.
     Thus, E_p must be placed in a stratum S_l more reliable than the stratum S_i that contains E_k, i.e., $p < i$. So, $i \neq 1$.
     Then, $\forall j$, $1 \leq j < i : S_j$ is built up from E_r's such that $|E_r \cap B| \leq |E_r \cap A|$ by construction of S_i.
     Hence $\forall j$, $1 \leq j < i : |S_j \cap B| \leq |S_j \cap A|$.
     In particular, $|S_l \cap B| < |S_l \cap A|$.
     Subsequently, we deduce that $A <_{lex} B$.
2. $A \approx_{\Delta} B$ implies $A =_{lex} B$:
   $\forall i$, $1 \leq i \leq m : S_i = \bigcup_{j \in X} E_j$ where $X \subseteq \{1, \ldots, n\}$.
   Then, $|S_i \cap A| = |(\bigcup_{j \in X} E_j) \cap A| = |\bigcup_{j \in X}(E_j \cap A)| = \sum_{j \in X} |E_j \cap A|$ since E_j's are disjoint.
   Similarly, we obtain $|S_i \cap B| = \sum_{j \in X} |E_j \cap B|$.
   According to Proposition 2, $A \approx_{\Delta} B$ iff $\forall i, 1 \leq i \leq n : |E_i \cap A| = |E_i \cap B|$.
   Hence, we conclude that $\forall j, 1 \leq j \leq m : |S_i \cap A| = |S_i \cap B|$, i.e., $A =_{lex} B$. ∎

Proposition 4
Let $(\Sigma, \preceq)$ be a partially preordered belief base and let ψ be a propositional formula. Then,

$$\textit{if } (\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi \textit{ then } (\Sigma, \preceq) \vdash^{\mathcal{C}}_{lex} \psi.$$

Proof.

By definition,

$$(\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi \text{ iff } \forall B \in \mathfrak{L}ex(\Sigma, \preceq) : B \vdash \psi$$

and

$$(\Sigma, \preceq) \vdash^{\mathcal{C}}_{Lex} \psi \text{ iff } \forall B \in \bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq) : B \vdash \psi$$

So, this amounts to prove that

$$\bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq) \subset \mathfrak{L}ex(\Sigma, \preceq)$$

- First, let us show that:
  $\bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq) \subseteq \mathfrak{L}ex(\Sigma, \preceq)$:

  Let $(\Sigma, \leq) \in \mathcal{C}(\Sigma, \preceq)$. Let $A \in Lex(\Sigma, \leq)$ and let us suppose that $A \notin \mathfrak{L}ex(\Sigma, \preceq)$.

  This means that there exists a consistent subbase B of Σ such that $B \prec_{\Delta} A$.

  Now, according to Lemma 1, we obtain $B <_{lex} A$. But we know that this is in conflict with the fact that $A \in Lex(\Sigma, \leq)$.

  Consequently, $A \in \mathfrak{L}ex(\Sigma, \preceq)$.
- We now show that:
  $\mathfrak{L}ex(\Sigma, \preceq) \nsubseteq \bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq)$:

  Indeed, let us consider the following counter example.

Example 6 *Let $(\Sigma, \preceq)$ be such that:*
$\Sigma = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2\}$ *with:*

– $\alpha_1 = a \wedge \neg b \wedge c$,
– $\beta_1 = \neg a \wedge \neg b \wedge c$,
– $\gamma_1 = b \wedge d$,
– $\alpha_2 = a \wedge \neg b \wedge d$,
– $\beta_2 = \neg a \wedge \neg b \wedge d$,
– $\gamma_2 = b \wedge e$,
– $\beta_3 = \neg a \wedge \neg b \wedge e$.

In addition, we have:

– $\alpha_1 \approx \gamma_1 \approx \gamma_2$ *and*
– $\alpha_2 \approx \beta_1 \approx \beta_2 \approx \beta_3$.

Clearly, $MCONS = \{A, B, C\}$ such that

– $A = \{\alpha_1, \alpha_2\}$,
– $B = \{\beta_1, \beta_2, \beta_3\}$,
– $C = \{\gamma_1, \gamma_2\}$.

On the one hand,

$$\mathfrak{L}ex(\Sigma, \preceq) = \{A, B, C\}.$$

In fact, we have:

– $A \sim_{\Delta} B$,
– $A \sim_{\Delta} C$,
– $B \sim_{\Delta} C$.

On the other hand,

$$\bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq) = \{B, C\}.$$

Hence, we deduce that

$$\mathfrak{L}ex(\Sigma, \preceq) \nsubseteq \bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq).$$

Consequently,

$$\bigcup_{(\Sigma,\leq)\in\mathcal{C}(\Sigma,\preceq)} Lex(\Sigma, \leq) \subset \mathfrak{L}ex(\Sigma, \preceq)$$

■

Proposition 5
$(\Sigma, \preceq) \vdash^{\mathcal{P}}_{lex} \psi$ *iff* $(\Sigma, \preceq) \models^{\mathcal{S}}_{lex} \psi$.

Proof.
Clearly enough, this amounts to prove that

$$\bigcup_{B\in\mathfrak{L}ex(\Sigma,\preceq)} Mod(B) = Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}}).$$

- Let us show that $\bigcup_{B\in\mathfrak{L}ex(\Sigma,\preceq)} Mod(B) \subseteq Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$:

  Let $\omega \in Mod(B)$ where $B \in \mathfrak{L}ex(\Sigma, \preceq)$ and let us suppose that $\omega \notin Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$.

  $\omega \notin Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$ iff $\exists \omega' \in \mathcal{W}$ such that $\omega' \prec^{\Delta}_{\mathcal{W}} \omega$, i.e., $[\omega'] \prec_{\Delta} [\omega]$.

  But we know that B is a maximal (with respect to set inclusion) coherent subbase so ω does no satisfy any formula in Σ outside B.

  Stated otherwise, $[\omega] = B$.

  Then, $B' \prec_{\Delta} B$ where $B' = [\omega']$.

  However, this is incoherent with the fact that $B \in \mathfrak{L}ex(\Sigma, \preceq)$.

  Thus, $\omega \in Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$.
- Now let us show that $Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}}) \subseteq \bigcup_{B\in\mathfrak{L}ex(\Sigma,\preceq)} Mod(B)$:

  Let $\omega \in Min(\mathcal{W}, \prec^{\Delta}_{\mathcal{W}})$.

  This means that there is no $\omega' \in \mathcal{W}$ such that $\omega' \prec^{\Delta}_{\mathcal{W}} \omega$.

  In particular, for any subbase $B \in \mathfrak{L}ex(\Sigma, \preceq)$, for any ω'' which is a model of B, $\omega'' \not\prec^{\Delta}_{\mathcal{W}} \omega$, i.e., $[\omega''] \not\prec_{\Delta} [\omega]$, but $[\omega"] = B$ since B is a maximal consistent subbase.

  Then, for any $B \in \mathfrak{L}ex(\Sigma, \preceq)$,$B \not\prec_{\Delta} [\omega]$.

  Consequently, $[\omega] \in \mathfrak{L}ex(\Sigma, \preceq)$.

  So, $\omega \in \bigcup_{B\in\mathfrak{L}ex(\Sigma,\preceq)} Mod(B)$. ■